

PROJECTIVE MANIFOLDS MODELED AFTER HYPERQUADRICS

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ABSTRACT. We classify projective manifolds with flat holomorphic conformal structures.

INTRODUCTION

Manifolds with certain reductions of the structure group are classical objects in real differential geometry. In the holomorphic setting, compact complex manifolds modeled after hermitian symmetric spaces have been studied by Kobayashi and Ochiai ([KoOc80], [KoOc81], [KoOc80]) and others, more recently by Mok and Hwang around rigidity questions ([HwMo97], [HwMo98]). In this article we give the complete list of projective manifolds with quadric structure.

Let M be a compact, connected complex manifold and S a compact hermitian symmetric space of the same complex dimension m . There is a notion of a *flat* S -structure and of an *infinitesimal* S -structure on M . If the universal covering space \tilde{M} of M admits an embedding $\tilde{M} \hookrightarrow S$, such that $\pi_1(M)$ becomes a subgroup of $\mathrm{HAut}(S)$, then M carries a flat S -structure. Typical examples are $M = S$, certain quotients of \mathbb{C}^m and compact quotients of the non compact dual of S in the sense of hermitian symmetric spaces. The infinitesimal version of an S -structure asks for a certain subbundle in the holomorphic frame bundle of M . Equivalent in the case of $S = \mathcal{Q}_m$, the m -dimensional hyperquadric, is the existence of a *holomorphic conformal structure*, i.e., a non-degenerate section

$$g \in H^0(M, S^2 T_M \otimes L), \quad L \in \mathrm{Pic}(M).$$

If M carries a quadric structure, then M carries a holomorphic conformal structure, called flat. Whether every holomorphic conformal structure is flat, i.e., comes from a quadric structure, is not known.

Any compact Riemann surface carries a flat holomorphic conformal structure by the uniformization theorem. Kobayashi and Ochiai gave a complete list in the compact surfaces case ([KoOc82]), proving flatness in dimension 2. The authors proved flatness of any holomorphic conformal structure in the case of projective threefolds ([JaRa05]). In the Kähler–Einstein case one has the following result ([KoOc82]):

0.1. Theorem. *A compact Kähler–Einstein manifold M_m carries a holomorphic conformal structure if and only if M is either*

- 1.) $\simeq \mathcal{Q}_m$,
- 2.) a torus or a conformally hyperelliptic manifold or

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3.) a \mathcal{D}_m quotient.

Conformally hyperelliptic manifolds are particular finite étale quotients of tori. The notion will be introduced in 1.5. A \mathcal{D}_m quotient means the universal covering space of M can be identified with \mathcal{D}_m , the non compact dual of \mathcal{Q}_m in the sense of hermitian symmetric spaces also reviewed in 1.4 below. We prove:

0.2. Theorem. *Let M_m be a projective manifold with a quadric structure which is not Kähler–Einstein. Then M is a complex surface and a fiber bundle over a compact Riemann surface C . More precisely, there exists a representation*

$$\rho : \pi_1(C) \longrightarrow \text{Aut}(F), \quad F \text{ rational or elliptic,}$$

such that $M = F \times_\rho C$.

Here $M = F \times_\rho C$ means the following. Let \tilde{C} denote the universal covering space of C . Then we obtain M by dividing $F \times \tilde{C}$ by the action of $\pi_1(C)$

$$\gamma(f, \tau) = (\rho(\gamma)(f), \gamma(\tau)), \quad \gamma \in \pi_1(C).$$

In other words, the only non–Kähler–Einstein examples arise in dimension 2, caused by the fact that $\mathcal{Q}_2 \simeq \mathbb{P}_1 \times \mathbb{P}_1$ is not irreducible as a hermitian symmetric space.

The above theorem reproves Kobayashi and Ochiai’s result in the case of projective surfaces as well as the author’s result in the case of projective threefolds and quadric structures. Some of the methods from the proof also apply to the non–flat case, however a general classification for manifolds with holomorphic conformal structures seems out of reach at the moment.

Notation. M denotes a compact, connected complex manifold of complex dimension m . We always assume M to be Kähler.

The holomorphic tangent bundle of M is T_M , its dual Ω_M^1 , the bundle of holomorphic 1–forms. The canonical bundle is $K_M = \det \Omega_M^1$. We do not distinguish between Cartier divisors and line bundles. For a line bundle write rL instead of $L^{\otimes r}$. In the case M projective we write \equiv for numerical equivalence of line bundles, i.e. equality of Chern classes in $H^1(M, \Omega_M^1)$. A fibration $M \longrightarrow N$ is holomorphic and has connected fibers.

1. THE COMPLEX PROJECTIVE HYPERQUADRIC AND QUADRIC STRUCTURES

There is a general notion of an S –structure, S an arbitrary hermitian symmetric space of the compact type. We only need the definition for the spacial case $S = \mathcal{Q}_m$, the m –dimensional complex projective quadric, and later for $S = \mathbb{P}_m(\mathbb{C})$. The general constructions are also explained in [KoOc82], [KoNo63], we include it for the convenience of the reader.

1.1. The hyperquadric $\mathcal{Q}_m \subset \mathbb{P}_{m+1}(\mathbb{C})$. Let $q = (q_{ij}) \in M_{m+2, m+2}(\mathbb{C})$ be symmetric and non–degenerate. The *complex projective quadric*

$$(1.1) \quad \mathcal{Q}(q) = \{[Z] = [Z_0 : \cdots : Z_{m+1}] \in \mathbb{P}_{m+1}(\mathbb{C}) \mid \sum q_{ij} Z_i Z_j = 0\}$$

is a smooth projective homogeneous variety of complex dimension m . Write $\mathcal{Q}_m = \mathcal{Q}(I_{m+2})$, where I_{m+2} is the unit matrix, for short. It is well known that $\mathcal{Q}_m \simeq$

$\mathcal{Q}(q)$ for any q by a projective linear change of coordinates. We will use these isomorphisms without further mentioning below. Recall

$$\mathcal{Q}_1 \simeq \mathbb{P}_1(\mathbb{C}) \quad \text{and} \quad \mathcal{Q}_2 \simeq \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}).$$

For $m > 2$, \mathcal{Q}_m is Fano with Picard number one. The quadric \mathcal{Q}_m is a hermitian symmetric space of type BDI . For the later definition of the non-compact dual we have to recall some of the background: the natural S^1 -bundle on \mathcal{Q}_m

$$(1.2) \quad \{M \in M_{m+2,2}(\mathbb{R}) \mid M^t M = I_2\} \longrightarrow \mathcal{Q}_m, \quad M \mapsto M \begin{pmatrix} i \\ 1 \end{pmatrix},$$

identifies $M, M' \in M_{m+2,2}(\mathbb{R})$ iff $M = M'D$ for some $D \in SO(2) \simeq S^1$. The columns of such an M define a ON-basis for the standard metric of a twoplane in \mathbb{R}^{m+2} . Geometrically, $M, M' \in M_{m+2,2}(\mathbb{R})$ are identified iff they span the same twoplane and with the same orientation. In other words

$$\mathcal{Q}_m = \mathbb{O}\mathbb{G}(2, m+2),$$

the Grassmannian of oriented twoplanes in \mathbb{R}^{m+2} . Recall that with the usual Grassmannian $\mathbb{G}(2, m+2)$ one allows $D \in O(2)$ which gives an étale $2 : 1$ covering $\mathbb{O}\mathbb{G}(2, m+2) \longrightarrow \mathbb{G}(2, m+2)$. Identifying $\mathcal{Q}_m \simeq \mathbb{O}\mathbb{G}(2, m+2)$, one has to divide by the fixed point free action of complex conjugation.

The S_1 -bundle in (1.2) is compatible with the natural transitive action of the orthogonal group $SO(m+2)$ on both sides by multiplication from the left. Stabilizer of the reference point $(0_{2,m}, I_2)^t$ or $[0 : \dots : 0 : i : 1] \in \mathcal{Q}_m$, respectively, is $SO(m) \times SO(2)$. Hence

$$\mathcal{Q}_m = \mathbb{O}\mathbb{G}(2, m+2) \simeq SO(m+2)/SO(m) \times SO(2).$$

We obtain the description of \mathcal{Q}_m as quotient of the connected compact simple real Lie group $SO(m+2)$ by the maximal connected proper subgroup $SO(m) \times SO(2)$ with non discrete center

$$Z(SO(m) \times SO(2)) = \begin{cases} SO(m) \times SO(2) & m = 1, 2 \\ \{I_m\} \times SO(2) & m > 2 \end{cases}$$

The real tangent bundle on $SO(m+2)/SO(m) \times SO(2)$ is induced by the adjoint representation of $SO(m) \times SO(2)$ on $\mathfrak{so}(m+2)/\mathfrak{so}(m) \times \mathfrak{so}(2)$. The complex structure on \mathcal{Q}_m is defined by $I_m \times D_{\frac{\pi}{2}} \in Z(SO(m) \times SO(2))$, where $D_\alpha \in SO(2)$ denotes the standard positive rotation matrix. The square of the complex structure defines the reflecting automorphism at the reference points. The group of holomorphic automorphisms of \mathcal{Q}_m is $H\text{Aut}(\mathcal{Q}(S)) \simeq PSO(m+2, \mathbb{C})$.

1.2. Quadric structures. M_m carries a *holomorphic quadric structure*, if there exists a holomorphic atlas $\{U_i, \varphi_i\}_{i \in I}$, such that

- 1.) $\varphi_i : U_i \hookrightarrow \mathcal{Q}_m$ is a holomorphic embedding for $i \in I$ and
- 2.) $\varphi_i \circ \varphi_j^{-1}$, defined on $\varphi_j(U_{ij}) \subset \mathcal{Q}_m$, is the restriction of some automorphism from $H\text{Aut}(\mathcal{Q}_m) = PSO(m+2, \mathbb{C})$.

Clearly \mathcal{Q}_m carries a quadric structure. If M carries a quadric structure and $M' \longrightarrow M$ is étale, then M' carries a quadric structure. A major source of examples is

1.3. Lemma. *If there exists an embedding of the universal covering space $\tilde{M} \hookrightarrow \mathcal{Q}_m$, such that $\pi_1(M)$ acts on \tilde{M} by restricting automorphisms from $H\text{Aut}(\mathcal{Q}_m)$, then M carries a quadric structure.*

The map $SO(m+2, \mathbb{C}) \rightarrow PSO(m+2, \mathbb{C})$ is $2 : 1$ étale. The line bundle $\mathcal{O}_{\mathcal{Q}_m}(d) = \mathcal{O}_{\mathbb{P}_{m+2}(\mathbb{C})}(d)|_{\mathcal{Q}_m}$ is $SO(m+2, \mathbb{C})$ -homogeneous, and $PSO(m+2, \mathbb{C})$ -homogeneous in the case d even. Hence, for d even, $\mathcal{O}_{\mathcal{Q}_m}(d)$ induces a line bundle on any manifold M with a quadric structure.

1.3. Development. ([KoOc80], [KoWu83]) Let M_m carry a quadric structure, let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas as above. Choose one coordinate chart (U_0, φ_0) . Given a point $p \in M_m$, choose a chain of charts $(U_i, \varphi_i)_{i=0, \dots, r}$ such that $U_i \cap U_{i-1} \neq \emptyset$, $i = 1, \dots, r$ and $p \in U_r$. We have

$$\varphi_{i-1} \circ \varphi_i^{-1} = g_i|_{\varphi_i(U_i \cap U_{i-1})}$$

for some $g_i \in HAut(\mathcal{Q}_m)$. Set $\psi(p) := (g_1 \circ g_2 \circ \dots \circ g_r \circ \varphi_r)(p) \in \mathcal{Q}_m$. This gives a multivalued map $\psi : M \rightarrow \mathcal{Q}_m$ which is single valued in the case M simply connected. It is called a development of M to \mathcal{Q}_m . By construction, ψ is immersive.

If M is compact and simply connected, then $M \simeq \mathcal{Q}_m$. If M is not simply connected, then let \tilde{M} be its universal covering space and let $\psi : \tilde{M} \rightarrow \mathcal{Q}_m$ be a development of \tilde{M} . We get a natural map $\rho : \pi_1(M) \rightarrow HAut(\mathcal{Q}_m)$ such that $\psi(\gamma(p)) = \rho(\gamma)(\psi(p))$ for any $\gamma \in \pi_1(M)$ ([KoWu83]).

1.4. Dual space \mathcal{D}_m . In a fixed basis of \mathbb{R}^{m+2} , let $I_{m,2}$ be the symmetric form of signature $(m \times -, 2 \times +)$. The real Lie group

$$SO(m, 2) = \{A \in Sl_{m+2}(\mathbb{R}) \mid A^t I_{m,2} A = I_{m,2}\}$$

is well known to have two connected components, we denote the one containing the identity $SO^0(m, 2)$. The symmetric space $BDI_{m,2}$ dual to $\mathbb{O}\mathbb{G}(2, m+2)$ parametrizes twoplanes in \mathbb{R}^{m+2} on which $I_{m,2} > 0$ is positive definit. Hence

$$BDI_{m,2} = \{M \in M_{m+2,2}(\mathbb{R}) \mid M^t I_{m,2} M > 0\} / \sim,$$

where $M \sim M'$ iff $M = M'D$ for some $D \in Gl_2(\mathbb{R})$. The twoplanes are generated by the columns of M . The group $SO(m, 2)^0$ acts transitively on $BDI_{m,2}$ from the left, stabilizer of $(0_{m,2}, I_2)^t$ is $SO(m, 2)^0 \cap S(O(m) \times O(2)) = SO(m) \times SO(2)$ and

$$BDI_{m,2} \simeq SO(m, 2)^0 / SO(m) \times SO(2).$$

If we write $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, $M_1 \in M_{m,2}(\mathbb{R})$, then $M^t I_{m,2} M > 0$ implies $M_2 \in Gl_2(\mathbb{R})$. As $M \sim MM_2^{-1}$ we find the classical description

$$BDI_{m,2} \simeq \{m \in M_{m,2}(\mathbb{R}) \mid m^t m < I_2\},$$

with the action of $SO(m, 2)^0$ given by

$$m \mapsto (Am + B)(Cm + D)^{-1}, \quad \begin{pmatrix} A_{m,m} & B_{m,2} \\ C_{2,m} & D_{2,2} \end{pmatrix} \in SO(m, 2)^0.$$

By choosing an ON-Basis, any twoplane on which $I_{m,2} > 0$ is represented by $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} \in M_{m+2,2}(\mathbb{R})$ such that $M^t I_{m,2} M = I_2$ and $\det M_2 > 0$. In this description, $M, M' \in M_{m+2,2}(\mathbb{R})$ define the same twoplane iff there exists $D \in SO(2)$ such that $M = M'D$ and $M \mapsto M_i = M \begin{pmatrix} i \\ 1 \end{pmatrix}$ gives

$$BDI_{m,2} \hookrightarrow \mathcal{Q}(I_{m,2}).$$

This identifies $BDI_{m,2}$ and the open subset of $\mathcal{Q}(I_{m,2})$

$$\mathcal{D}_m = \{[W] \in \mathcal{Q}(I_{m,2}) \mid W^t I_{m,2} \bar{W} > 0, i(\bar{W}_{m+1} W_{m+2} - W_{m+1} \bar{W}_{m+2}) > 0\}.$$

The group of holomorphic automorphisms of \mathcal{D}_m is

$$\mathrm{HAut}(\mathcal{D}_m) = PSO^0(m, 2) = \{f \in \mathrm{HAut}(\mathcal{Q}(I_{m+2})) \mid f(\mathcal{D}_m) = \mathcal{D}_m\}.$$

By Lemma 1.3,

1.4. Proposition. *Any complex manifold whose universal covering space is \mathcal{D}_m carries a quadric structure.*

In order to see that \mathcal{D}_m is a bounded symmetric domain let $Z_{m+1} = \frac{1}{\sqrt{2}}(W_{m+1} - iW_{m+2})$, $Z_{m+2} = \frac{1}{\sqrt{2}}(W_{m+1} + iW_{m+2})$ and $Z_k = W_k$ for $k = 1, \dots, m$. In these new coordinates the defining equation for $\mathcal{Q}(I_{m,2})$ becomes

$$2Z_{m+1}Z_{m+2} = Z_1^2 + \dots + Z_m^2,$$

i.e. the change of coordinates transforms $\mathcal{Q}(I_{m,2})$ into $\mathcal{Q}(q_0)$, where

$$q_0 = \begin{pmatrix} I_m & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

The intersection $\mathcal{Q}(q_0) \cap \{Z_{m+2} \neq 0\}$ is isomorphic to \mathbb{C}^m by

$$j : \mathbb{C}^m \longrightarrow \mathcal{Q}(q_0), \quad z \mapsto [z : \frac{1}{2}z^t z : 1].$$

Now $\mathcal{D}_m = \{[Z] \in \mathcal{Q}(q_0) \mid Z^t I_{m,2} \bar{Z} > 0, |Z_{m+1}|^2 < |Z_{m+2}|^2\} \subset \mathcal{Q}(q_0) \cap \{Z_{m+2} \neq 0\}$ and one checks $\mathcal{D}_m = j(\{z \in \mathbb{C}^m \mid z^t \bar{z} < 1 + |\frac{1}{2}z^t z|^2 < 2\})$.

1.5. Conformally hyperelliptic varieties. (see also [KoOc82], [KoWu83]) Let $CO\mathrm{Aff}_m(\mathbb{C})$ be the group of conformal affine maps

$$\sigma_{A,b,c} : \mathbb{C}^m \longrightarrow \mathbb{C}^m, \quad \sigma_{A,b,c}(z) = cAz + b, \quad A \in O(m, \mathbb{C}), b \in \mathbb{C}^m, c \in \mathbb{C}^*.$$

A direct computation shows $\rho : CO\mathrm{Aff}_m(\mathbb{C}) \longrightarrow O(q_0, \mathbb{C})$ given by

$$\rho(\sigma_{A,b,c}) = \begin{pmatrix} A & 0_{m,1} & \frac{1}{c}b \\ b^t A & c & \frac{1}{2c}b^t b \\ 0_{1,m} & 0 & \frac{1}{c} \end{pmatrix}$$

is a group morphism and $j(\sigma_{A,b,c}(z)) = \rho(\sigma_{A,b,c})(j(z))$ with j from above.

We will call a compact complex manifold M *conformally hyperelliptic*, if M is not a torus but isomorphic to a quotient of \mathbb{C}^m by some fixed point free subgroup $\pi_1 \subset CO\mathrm{Aff}_m(\mathbb{C})$ which contains a complete lattice Λ as a subgroup of finite index.

Such an M is a finite étale quotient of a torus. By Lemma 1.3, using j and ρ we have

1.5. Proposition. *Tori and conformally hyperelliptic manifolds carry a quadric structure.*

2. HOLOMORPHIC CONFORMAL STRUCTURES

A complex manifold M is said to carry a *holomorphic conformal structure* if there exists a non degenerate holomorphic

$$g \in H^0(M, S^2\Omega_M \otimes L^*),$$

where $L \in \text{Pic}(M)$ is some line bundle. An equivalent definition involves a $CO(m, \mathbb{C}) = \{cA \mid A \in O(m, \mathbb{C}), c \in \mathbb{C}^*\}$ bundle in the frame bundle of M , see [KoOc82].

We will use the same letter g to denote the induced map from $T_M \otimes L^*$ to Ω_M^1 by g . By assumption g gives an isomorphism

$$(2.1) \quad g : T_M \otimes L^* \simeq \Omega_M^1 \quad \text{and} \quad -2K_M \simeq mL.$$

On the quadric $\mathcal{Q}(q)$ in (1.1), $\sum_{ij} q_{ij} dZ_i dZ_j$ induces a holomorphic conformal structure with the choice $L = \mathcal{O}_{\mathbb{P}_{m+1}(\mathbb{C})}(-2)|_{\mathcal{Q}_m}$. It is invariant under the action of $\text{HAut}(\mathcal{Q}_m)$. A manifold M with a quadric structure carries a holomorphic conformal structure. We will call the conformal structure *flat* in this case.

2.1. Atiyah class of Ω_M^1 . The *Atiyah class* $b(E) \in H^1(M, \text{Hom}(E, E) \otimes \Omega_M^1)$ associated to a holomorphic vector bundle E on a complex manifold M ([At57]) is the obstruction to the holomorphic splitting of the first jet sequence

$$0 \longrightarrow E \otimes \Omega_M^1 \longrightarrow J_1(E) \longrightarrow E \longrightarrow 0.$$

Let $\mathfrak{U} = \{U_\alpha\}$ be a trivializing atlas, i.e., $E|_{U_\alpha} \simeq \mathbb{C}^r \times U_\alpha =: E_\alpha$. Denote the induced transition functions by $t_{\alpha\beta} : E_\beta \longrightarrow E_\alpha$, defined over $U_{\alpha\beta}$, i.e., $t_{\alpha\beta} \in GL_m(\mathcal{O}_{U_{\alpha\beta}})$. Then the matrix of holomorphic oneforms $dt_{\alpha\beta}$ is a well defined 1-cocycle of $\text{Hom}(E, E) \otimes \Omega_M^1$, representing $b(E)$. Under Dolbeault $H^1(M, \text{Hom}(E, E) \otimes \Omega_M^1) \simeq H^{1,1}(M, \text{Hom}(E, E))$, $b(E)$ corresponds to $[\Theta_h]$, where Θ_h is the curvature tensor associated to some hermitian metric h on E . In the Kähler case the first Chern class of E is given by $\text{tr}(b(E)) = -\frac{1}{2i\pi}c_1(E) \in H^1(M, \Omega_M^1)$. This is why we normalize $a(E) := -\frac{1}{2i\pi}b(E)$.

Consider $E = T_M$. Let $\mathfrak{U} = \{(U_\alpha, z_\alpha^1, \dots, z_\alpha^m)\}$ be an atlas. Transition functions of T_M are $t_{\alpha\beta} = (\frac{\partial z_\alpha^i}{\partial z_\beta^q})_{i,q}$ and $b(T_M)$ is represented by

$$\sum_{iqr} \frac{\partial^2 z_\alpha^i}{\partial z_\beta^q \partial z_\beta^r} \frac{\partial}{\partial z_\alpha^i} \otimes dz_\beta^q \otimes dz_\beta^r = \sum_{ipqr} \frac{\partial^2 z_\alpha^i}{\partial z_\beta^q \partial z_\beta^r} \frac{\partial z_\beta^p}{\partial z_\alpha^i} \frac{\partial}{\partial z_\beta^p} \otimes dz_\beta^q \otimes dz_\beta^r.$$

Let M be compact Kähler with a holomorphic conformal structure. Then $c_1(K_M) \in H^1(M, \Omega_M^1)$ and via $g : T_M \otimes L^* \longrightarrow \Omega_M^1$

$$g^{-1}(c_1(K_M)) \in H^1(M, T_M \otimes L^*).$$

Then $g \otimes g^{-1}(c_1(K_M)) \in H^1(M, S^2\Omega_M \otimes L \otimes T_M \otimes L^*) \simeq H^1(M, S^2\Omega_M \otimes T_M)$. The following Proposition is a consequence of standard formulas from Riemannian geometry interpreted in the holomorphic setting.

2.2. Proposition. *On a compact Kähler manifold with a holomorphic conformal structure $g \in H^0(M, S^2\Omega_M^1 \otimes L)$, the normalized Atiyah class of Ω_M^1 is given by*

$$(2.3) \quad a(\Omega_M^1) = \frac{c_1(K_M)}{m} \otimes id + id \otimes \frac{c_1(K_M)}{m} - g \otimes g^{-1}\left(\frac{c_1(K_M)}{m}\right)$$

in $H^1(M, \Omega_M^1 \otimes T_M \otimes \Omega_M^1)$.

Note $\Omega_M^1 \otimes T_M \otimes \Omega_M^1 \simeq \Omega_M^1 \otimes \text{End}(T_M) \simeq \text{End}(T_M) \otimes \Omega_M^1$. Proposition 2.2 implies certain relations among the Chern classes of M . For example

$$(2.4) \quad m^2 c_2(M) = \left(\binom{m}{2} + 1 \right) c_1^2(M) - H^2(M, \Omega_M^2).$$

Roughly speaking any algebraic relation among Chern classes on the quadric holds on M . For details see [KoOc82].

Proof. Choose an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha^1, \dots, z_\alpha^m)\}$ such that, locally, g is given by

$$g_\alpha = \sum_{i,j} g_{\alpha ij} dz_\alpha^i \otimes dz_\alpha^j, \quad g_{\alpha ij} \in \mathcal{O}_M(U_\alpha)$$

with $g_{\alpha ij} = g_{\alpha ji}$ and $g_\alpha = l_{\alpha\beta} g_\beta$. Here $l_{\alpha\beta} \in \mathcal{O}_{U_{\alpha\beta}}^*$ is a 1-cocycle defining L . As a map $T_M \rightarrow \Omega_M^1 \otimes L^*$ in the chosen frame, g is represented by the matrix $g_{\alpha ij}$, the inverse is denoted (g_α^{ij}) . As $2K_M = mL$, $\frac{1}{2} \sum_h \frac{\partial l_{\alpha\beta}}{\partial z_\beta^h} dz_\beta^h$ represents $c_1(K_M)/m$ in $H^1(M, \Omega_M^1)$. We have to show that there is a cocycle solution to

$$\sum_{pqr} \left(\sum_i \frac{\partial z_\beta^p}{\partial z_\alpha^i} \frac{\partial^2 z_\alpha^i}{\partial z_\beta^q \partial z_\beta^r} - \frac{1}{2} \delta_{pq} \frac{\partial l_{\alpha\beta}}{\partial z_\beta^r} - \frac{1}{2} \delta_{pr} \frac{\partial l_{\alpha\beta}}{\partial z_\beta^q} + \frac{1}{2} g_\beta^{ph} \frac{\partial l_{\alpha\beta}}{\partial z_\beta^h} g_{\beta qr} \right) \frac{\partial}{\partial z_\beta^p} \otimes dz_\beta^q \otimes dz_\beta^r.$$

This is a consequence of standard formulas from Riemannian geometry: as in the Riemannian case we associate to g_α a holomorphic affine connection on U_α by

$$(2.5) \quad \Gamma_{jk}^i = \Gamma_{jk}^i(g_\alpha) = \frac{1}{2} \sum_h g_\alpha^{ih} \left(\frac{\partial g_{\alpha hj}}{\partial z_\alpha^k} + \frac{\partial g_{\alpha hk}}{\partial z_\alpha^j} - \frac{\partial g_{\alpha jk}}{\partial z_\alpha^h} \right).$$

Let $\tilde{g}_\beta = l_{\alpha\beta} g_\beta$. Then $g_\alpha = \tilde{g}_\beta$ on $U_{\alpha\beta}$ as in the Riemannian case. Therefore, if we put $\tilde{\Gamma}_{qr}^p = \Gamma_{qr}^p(\tilde{g}_\beta)$, then we have the well known formula of base change for the Christoffel symbols:

$$\tilde{\Gamma}_{\beta qr}^p = \sum_{ijk} \frac{\partial z_\alpha^j}{\partial z_\beta^q} \frac{\partial z_\alpha^k}{\partial z_\beta^r} \Gamma_{\alpha jk}^i \frac{\partial z_\beta^p}{\partial z_\alpha^i} + \sum_i \frac{\partial z_\beta^p}{\partial z_\alpha^i} \frac{\partial^2 z_\alpha^i}{\partial z_\beta^q \partial z_\beta^r}.$$

Using product rule and $g_\beta g_\beta^{-1} = id$ we may express $\tilde{\Gamma}_{qr}^p$ in terms of $\Gamma_{qr}^p(g_\beta)$:

$$\tilde{\Gamma}_{\beta qr}^p = \frac{1}{2} \sum_h l_{\alpha\beta}^{-1} \left(\delta_{pq} \frac{\partial l_{\alpha\beta}}{\partial z_\beta^r} + \delta_{pr} \frac{\partial l_{\alpha\beta}}{\partial z_\beta^q} - g_\beta^{ph} \frac{\partial l_{\alpha\beta}}{\partial z_\beta^h} g_{\beta qr} \right) + \Gamma_{qr}^p,$$

where $\Gamma_{qr}^p = \Gamma_{qr}^p(g_\beta)$. Then $\sum_{ijk} \Gamma_{jk}^i \frac{\partial}{\partial z_\alpha^i} \otimes dz_\alpha^j \otimes dz_\alpha^k \in Z(\mathfrak{U}, T_M \otimes \Omega_M^1 \otimes \Omega_M^1)$ is the desired cocycle solution. \square

2.2. Extensions. Let Ω be an arbitrary vector bundle on some complex manifold M . Let $L \in \text{Pic}(M)$ be some line bundle and $g \in H^0(M, S^2\Omega \otimes L)$. Assume there is a vector bundle F , $rk F > 0$, and an exact sequence

$$(2.6) \quad 0 \rightarrow F \rightarrow \Omega \rightarrow F^* \otimes L^* \rightarrow 0.$$

Dualize and multiply by L^* . We say g is compatible with (2.6), if we obtain a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \Omega^* \otimes L^* & \longrightarrow & F^* \otimes L^* & \longrightarrow & 0 \\ & & \downarrow \varphi & & \downarrow g & & \downarrow \psi & & \\ 0 & \longrightarrow & F & \longrightarrow & \Omega & \longrightarrow & F^* \otimes L^* & \longrightarrow & 0. \end{array}$$

Obviously, g is compatible in the case $\text{Hom}(F, F^* \otimes L) = 0$. A compatible $g \in H^0(M, S^2\Omega \otimes L)$ is non-degenerate if and only if φ and ψ are isomorphisms.

2.7. Lemma. *Let $e \in H^1(M, F \otimes F \otimes L)$ be the extension class of (2.6). Assume $\text{Hom}(F, F) \simeq \mathbb{C}$. Then there exists a non-degenerate $g \in H^0(M, S^2\Omega \otimes L)$ compatible with (2.6) if and only if $e \in H^1(M, \bigwedge^2 F \otimes L)$.*

Here we use the canonical isomorphism $F \otimes F \simeq S^2F \oplus \bigwedge^2 F$. Note that in the particular case $rk F = 1$ the condition $e \in H^1(M, \bigwedge^2 F \otimes L)$ implies the holomorphic splitting of (2.6).

Proof. From (2.6) we obtain

$$\begin{aligned} 0 &\longrightarrow K \otimes L \longrightarrow S^2\Omega \otimes L \longrightarrow S^2F^* \otimes L^* \longrightarrow 0 \\ 0 &\longrightarrow S^2F \otimes L \longrightarrow K \otimes L \longrightarrow F \otimes F^* \longrightarrow 0. \end{aligned}$$

The set of compatible g 's is $H^0(M, K \otimes L) \subset H^0(M, S^2\Omega \otimes L)$. A section coming from $H^0(M, S^2F \otimes L)$ will be degenerated.

Since $\text{Hom}(F, F) \simeq \mathbb{C}$, a compatible non-degenerate g exists if and only if we can lift $id_F \in H^0(M, F \otimes F^*)$ to some $g \in H^0(M, K \otimes L)$. Such a g exists if and only if id_F is mapped to zero by the coboundary map $\delta : H^0(M, F \otimes F^*) \longrightarrow H^1(M, S^2F \otimes L)$. Now $\delta(id_F)$ is the image of e under $H^1(M, F \otimes F \otimes L) \longrightarrow H^1(M, S^2F \otimes L)$. Therefore g exists if and only if $e \in H^1(M, \bigwedge^2 F \otimes L)$. \square

3. HYPERELLIPTIC MANIFOLDS

A compact complex manifold M_m is called *hyperelliptic* if M is a finite étale quotient of a complex torus $T = \mathbb{C}^m/\Lambda$. The universal covering space of such an M can be identified with \mathbb{C}^m such that $\pi_1(M) \subset \text{HAut}(\mathbb{C}^m)$ can be identified with a subgroup of $\text{Aff}_m(\mathbb{C})$, the group of affine maps $\sigma_{A,b} : z \mapsto Az + b$, $A \in GL_m(\mathbb{C})$, $b \in \mathbb{C}^m$.

The following description can essentially be found in [KoOc82]:

3.1. Theorem. *The following conditions on a hyperelliptic manifold M_m are equivalent:*

- 1.) M carries a quadric structure.
- 2.) M carries a holomorphic conformal structure.
- 3.) M is conformally hyperelliptic.

Proof. 1.) \Rightarrow 2.) is true for any manifold, 3.) \Rightarrow 1.) is Proposition 1.5. We prove 2.) \Rightarrow 3.).

By assumption, M is the quotient of \mathbb{C}^m by some subgroup of $\text{Aff}_m(\mathbb{C})$ which contains a complete lattice Λ as a normal subgroup of finite index. The lattice defines a torus T and a finite étale Galois covering $\pi : T \longrightarrow M$. Let $g \in H^0(M, S^2\Omega_M^1 \otimes L)$

be the holomorphic conformal structure. Then $g_T = \pi^*g \in H^0(T, S^2\Omega_T^1 \otimes \pi^*L)$ is non degenerate. By (2.1) π^*L is torsion on T , because K_T is trivial. After some finite étale covering $T' \rightarrow T$ to another torus T' , L becomes trivial. Another covering is Galois, so we may assume that already $\pi^*L = \mathcal{O}_T$.

Then L is induced by some representation $\rho : \pi_1(M) \rightarrow \mathbb{C}^*$, which is trivial on Λ . We may choose coordinates such that g_T is induced by $dz_1^2 + \dots + dz_m^2$ on \mathbb{C}^m . Then g_T descends to a section of $S^2\Omega_M^1 \otimes L$ if and only if $\sigma_{A,b}^*\omega_T = \rho(\sigma_{A,b})\omega_T$. This is the case if and only if $\sigma_{A,b} \in COAff_m(\mathbb{C})$. \square

3.2. Example. 1.) Let E, F be elliptic curves. Let $U \subset E$ be some nontrivial subgroup and $\rho : U \rightarrow Aut(F)$ be injective. The group U acts on $E \times F$ without fixed points via $(z_1, z_2) \mapsto (z_1 + u, \rho(u)(z_2))$. The quotient $M = E \times F/U$ is a hyperelliptic surface with a holomorphic conformal structure induced by $dz_1 dz_2$.

2.) Let E_1, E_2 be elliptic curves, $F = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\xi$ where ξ is some third root of unity. Let $e_6 \in E_1[6]$ be a point of order 6. Consider the fixed point free action of \mathbb{Z}_6 on $T = E_1 \times E_2 \times F$:

$$(z_1, z_2, z_3) \mapsto (z_1 + e_6, -z_2, \xi z_3).$$

It is easy to see that every eigenvector of the induced action on $H^0(T, S^2\Omega_T^1)$ is degenerate. The quotient $M = E_1 \times E_2 \times F/\mathbb{Z}_6$ is a hyperelliptic manifold that does not admit a quadric structure.

4. CLASSIFICATION

The Theorem by Kobayashi and Ochiai from the introduction says a compact Kähler–Einstein manifold M_m carries a holomorphic conformal structure if and only if up to isomorphisms M is either

- 1.) \mathcal{Q}_m ,
- 2.) a torus or a conformally hyperelliptic manifold or
- 3.) a \mathcal{D}_m quotient.

Recall that Kähler–Einstein implies $\pm c_1(M) > 0$ or $c_1(M) = 0$. Conversely, by results of Aubin [Au76] and Yau [Ya78], if $c_1(M) < 0$ or $c_1(M) = 0$, then M admits a Kähler–Einstein metric.

4.1. Rational curves. Let M_m be a projective manifold with a quadric structure. If K_M is not nef, then M contains a rational curve by the cone theorem ([KoMo98]) meaning that there exists a non-constant holomorphic map

$$\nu : \mathbb{P}_1(\mathbb{C}) \rightarrow M.$$

4.1. Proposition. *Let M_m be a projective manifold with a quadric structure. If M contains a rational curve then M is uniruled and either*

- 1.) $M \simeq \mathcal{Q}_m$ or
- 2.) M is ruled surface over a compact Riemann surface B of genus $g_B > 0$ induced by some representation $\rho : \pi_1(B) \rightarrow PGL_2(\mathbb{C})$.

In the second point, $M = \mathbb{P}_1(\mathbb{C}) \times \tilde{B}/\pi_1(B)$, where $\gamma \in \pi_1(B)$ acts via $(\tau, b) \mapsto (\rho(\gamma)(\tau), \gamma(b))$.

4.2. Corollary. *Let M_m be a projective manifold with a quadric structure. If M does not contain a rational curve, then K_M is nef.*

Recall that K_M nef means $K_M \cdot C \geq 0$ for any curve C in M . The corollary is an immediate consequence of the proposition and the cone theorem ([KoMo98]).

Proof of Proposition 4.1. Let $\psi : \tilde{M} \rightarrow Q_m$ be a development of the universal covering space $\mu : \tilde{M} \rightarrow M$. We have an induced map $\tilde{\nu} : \mathbb{P}_1(\mathbb{C}) \rightarrow \tilde{M}$ such that $\mu \circ \tilde{\nu} = \nu$. Homogeneity implies T_{Q_m} is spanned by global sections. Then $\nu^*T_M = (\psi \circ \tilde{\nu})^*T_{Q_m}$ is nef, i.e., $\nu^*T_M = \mathcal{O}_{\mathbb{P}_1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_1}(a_m)$ where $a_k \geq 0$ for all k . Then M contains a free rational curve. Then M is uniruled, i.e., covered by rational curves, by [Ko96], IV, 1.9.Theorem.

In the case $m \neq 2$ the quadric Q_m is irreducible as a hermitian symmetric space. Then M uniruled implies $M \simeq Q_m$ by [HwMo97].

In the case $m = 2$, M is a projective surface. As T_M is nef on every rational curve, M is free of (-1) -curves. The list of minimal uniruled projective surfaces is \mathbb{P}_2 or a ruled surface over some smooth base curve B .

As $9 = c_1(\mathbb{P}_2)^2 = 3c_2(\mathbb{P}_2)$ implying $c_2(\mathbb{P}_2) \neq \frac{1}{2}c_1(\mathbb{P}_2)^2$ we infer $M \neq \mathbb{P}_2$. On a ruled surface $\pi : M \rightarrow B$ we have the exact sequence

$$(4.3) \quad 0 \rightarrow \pi^*K_B \rightarrow \Omega_M^1 \rightarrow K_{M/B} \rightarrow 0.$$

The section g is necessarily compatible with this extension, since $\text{Hom}(\pi^*K_B, \pi^*T_B \otimes K_M) = H^0(B, T_B^{\otimes 2} \otimes \pi_*K_M) = 0$. By Lemma 2.7 the sequence splits holomorphically. This means the fiber bundle $M \rightarrow B$ has a connection. Then M is induced by a representation of the fundamental group $\pi_1(B) \rightarrow \text{Aut}(\mathbb{P}_1) = \text{PGL}_2(\mathbb{C})$. \square

4.2. Large fundamental group. Recall from ([Ko93], 1.7.) that M is said to have *large fundamental group* if for any irreducible complex subvariety $W \subset M$ of positive dimension

$$\text{Im}[\pi_1(W_{\text{norm}}) \rightarrow \pi_1(M)] \text{ is infinite.}$$

Here W_{norm} denotes the normalization of W .

4.4. Proposition. *A projective manifold M with a quadric structure which is not uniruled has large fundamental group.*

Proof. If M is not uniruled K_M is nef (Corollary 4.2). Assume to the contrary $\text{Im}[\pi_1(W_{\text{norm}}) \rightarrow \pi_1(M)]$ is finite for some W as above. Denote the normalization map by $\nu : W_{\text{norm}} \rightarrow W$.

Let C_{norm} be some general curve in W_{norm} , i.e., the intersection of $\dim W - 1$ general hyperplane sections. Think of C_{norm} as the normalization of $C := \nu(C_{\text{norm}}) \subset W$. We have

$$\pi_1(C_{\text{norm}}) \rightarrow \pi_1(W_{\text{norm}}) \rightarrow \pi_1(M).$$

Then $\text{Im}[\pi_1(C_{\text{norm}}) \rightarrow \pi_1(M)]$ is finite. The kernel of $\pi_1(C_{\text{norm}}) \rightarrow \pi_1(M)$ induces a finite étale covering $C' \rightarrow C_{\text{norm}}$ from a compact Riemann surface C' , such that $\mu : C' \rightarrow C \subset M$ factors over \tilde{M} , the universal covering space of M . Let $\psi : \tilde{M} \rightarrow Q_m$ be a development. Denote the induced map $C' \rightarrow \tilde{M} \rightarrow Q_m$ by ψ_1 . Then $\mu^*T_M = \psi_1^*T_{Q_m}$, implying $\deg(\mu^*(-K_M)) = \deg \psi_1^*(-K_{Q_m}) > 0$. This contradicts K_M nef. \square

4.3. Minimal case. Recall the following basic facts from the minimal model program: let M be a *minimal* projective manifold, i.e., assume K_M nef. The *abundance conjecture* asserts that $|dK_M|$ is spanned for some $d \gg 0$, defining the *Iitaka fibration*

$$\Phi : M \longrightarrow Y,$$

onto some normal projective Y of dimension $\dim Y = \kappa(M)$. Here κ denotes the *Kodaira dimension*, Φ has connected fibers. The abundance conjecture is known to hold true for $\dim M \leq 3$ and in the case K_M big (meaning $K_M^{\dim M} > 0$) by the base point free theorem. It is open in higher dimensions.

If M does not contain any rational curve, then M is minimal by the cone theorem ([KoMo98]). Moreover, in that case any rational map

$$M' \dashrightarrow M$$

from a manifold M' must be holomorphic. Indeed, if we first had to blow up M' in order to make it holomorphic, then we would find a rational curve in M . A manifold with this property is sometimes called *strongly minimal* or *absolutely minimal* ([Mori87], §9).

4.5. Theorem. *Let M be a projective manifold with a quadric structure. Assume that M does not contain a rational curve. Then there exists a finite étale covering $M' \rightarrow M$, such that M' is an abelian group scheme over some projective manifold N' such that $K_{N'}$ is ample.*

$$\begin{array}{ccc} M' & \longrightarrow & M \\ \downarrow & & \\ N' & & \end{array}$$

Here M' an abelian group scheme over N' means there exists a smooth surjective holomorphic map $f : M' \rightarrow N'$ such that every fiber is an abelian variety. Moreover, f admits a section.

Proof. The proof is essentially as in the case of a flat projective structure ([JaRa15], §3), we therefore only sketch the idea.

Since M does not contain any rational curve, K_M is nef. The quadric structure gives a non-trivial representation of $\pi_1(M)$. We obtain a non-trivial rational map $M \dashrightarrow Y$ with Y of general type ([Zu96], [Ka85]). Using [Lai11] we conclude K_M abundant.

Let $\Phi : M \rightarrow Y$ be the Iitaka fibration with general fiber F . Then $N_{F/M} \simeq \mathcal{O}_F^{\oplus \dim M - \dim Y}$, implying $K_F = K_M|_F + \det N_{F/M} = K_M|_F$ is torsion in $\text{Pic}(F)$. It induces a finite étale covering $\tilde{F} \rightarrow F$ such that $K_{\tilde{F}}$ is trivial.

By Beauville's decomposition theorem ([Bo74], [Be83]) we have, perhaps after another étale covering,

$$\tilde{F} \simeq A_y \times B_y$$

where A_y is abelian and B_y is simply connected. By Proposition 4.4, $\pi_1(B_y) = \{id\}$ implies B_y is a point. Therefore, F is an étale quotient of an abelian variety.

By ([Ko93], 6.3. Theorem), there exists a finite étale covering $M' \rightarrow M$, s.t. M' is *birational* to an abelian group scheme over some base Y' obtained as Stein factorization of $M' \rightarrow M \rightarrow Y$.

Since M' cannot contain a rational curve, it is strongly minimal and the birational map must be an isomorphism. Then $M' \rightarrow Y'$ is an abelian group scheme. The map has a section. Then Y' does not contain a rational curve. Then $K_{Y'}$ is nef.

By [Ko93], Proposition 5.9., $\kappa(M') = \kappa(Y')$. For any line bundle L on a projective manifold, $\kappa(L) \leq \nu(L) := \min\{k \mid L^k \equiv 0\}$. Then $\nu(K_{Y'}) \geq \kappa(Y') = \kappa(M') = \dim Y'$ implies $K_{Y'}$ big. By the base point free theorem ([KoMo98]), $|dK_{Y'}|$ is spanned for some $d \gg 0$, inducing the Iitaka fibration $Y' \rightarrow Z$ which is birational in this case. Fibers of positive dimension are covered by rational curves by [Ka91], Theorem 2. Then $Y' \rightarrow Z$ is finite and $K_{Y'}$ is ample. \square

5. ABELIAN GROUP SCHEMES

Because of Theorem 4.5 we now consider the following situation: $f : M \rightarrow N$ is an abelian group scheme, K_N is ample. We exclude the possibility of M being Kähler–Einstein by assuming

$$(5.1) \quad 0 < n = \dim_{\mathbb{C}} N < m = \dim_{\mathbb{C}} M.$$

The bundle of relative 1-forms $\Omega_{M/N}^1$ is the pullback of the rank $m - n$ bundle $E^{1,0} = f_*\Omega_{M/N}^1$, $E^{0,1} = R^1f_*\mathcal{O}_M \simeq E^{1,0*}$. We have the exact sequence

$$0 \rightarrow f^*\Omega_N^1 \rightarrow \Omega_M^1 \rightarrow f^*E^{1,0} \simeq \Omega_{M/N}^1 \rightarrow 0.$$

When restricted to N this is the dualized tangent sequence of N and $E^{1,0} = N_{N/M}^*$ under this identification. The connecting morphism

$$f_*f^*E^{1,0} \simeq E^{1,0} \rightarrow R^1f_*f^*\Omega_N^1 \simeq \Omega_N^1 \otimes R^1f_*\mathcal{O}_M \simeq \Omega_N^1 \otimes E^{0,1}$$

is the Kodaira Spencer map. We want to prove:

5.2. Proposition. *Let $M \rightarrow N$ be as above. Then M has a holomorphic conformal structure if and only if M is a complex surface and*

$$M \simeq F \times_{\rho} C$$

where F is an elliptic curve, C is a curve of positive genus and $\rho : \pi_1(C) \rightarrow \text{Aut}(F)$ is some representation.

The proof is given below. First:

5.3. Lemma. $L = f^*\mathcal{L}$ for a unique $\mathcal{L} \in \text{Pic}(N)$.

Restricted to the section N of f we have (in $H^1(\Omega^1)$):

$$(5.4) \quad \mathcal{L} \equiv L|_N \equiv -\frac{2}{m}K_M|_N \equiv -\frac{2}{m}(K_{M/N} + K_N).$$

Proof. For any fiber F of $M \rightarrow N$ we have $N_{F/M} \simeq \mathcal{O}_F^{\oplus m-n}$ and $\mathcal{O}_F = K_M|_F$ by adjunction. Because of (2.1) $L|_F$ is a torsion line bundle. Consider

$$(5.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & f^*\Omega_N^1 & \longrightarrow & \Omega_M^1 & \longrightarrow & f^*E^{1,0} \longrightarrow 0 \\ & & & & \downarrow g^{-1} & & \\ 0 & \longrightarrow & f^*E^{1,0*} \otimes L^* & \longrightarrow & T_M \otimes L^* & \longrightarrow & f^*T_N \otimes L^* \longrightarrow 0. \end{array}$$

Restrict to F . The top sequence and $f^*\Omega_N^1|_F \simeq \mathcal{O}_F^n$ shows $H^0(F, \Omega_M^1|_F) \neq 0$. The lower sequence shows $T_M \otimes L^*|_F$ is an extension of $L^{\oplus m-n}$ by $L^{\oplus n}$. Using $H^0(F, T_M|_F \otimes L) \simeq H^0(F, \Omega_M^1|_F) \neq 0$ we find $H^0(F, L|_F) \neq 0$.

A torsion line bundle with a section is trivial, i.e., $L|_F \simeq \mathcal{O}_F$. Then $\mathcal{L} := f_*L \in \text{Pic}(N)$ and $L = f^*\mathcal{L}$. Uniqueness follows from the injectivity of $\text{Pic}(N) \rightarrow \text{Pic}(M)$ ($M \rightarrow N$ has a section). Restricted to N we have $L|_N \simeq \mathcal{L}$. \square

5.1. Slope and stability. These notions depend on the choice of a polarization. We will only study bundles on N and will choose K_N as the polarization.

For any torsion free sheaf \mathcal{F} on N of positive rank, the slope is

$$\mu(F) = \frac{\det \mathcal{F}^{**} \cdot K_N^{n-1}}{rk \mathcal{F}}.$$

A bundle E is called (*semi-*) *stable* if for any proper nonzero subsheaf \mathcal{F} of E one has $\mu(\mathcal{F}) < (=)\mu(E)$. More generally a Higgs bundle (E, θ) is called (*semi-*) *stable* if for any proper nonzero Higgs-subsheaf \mathcal{F} one has $\mu(\mathcal{F}) < (=)\mu(E)$.

For example, Yau's uniformization theorem, nicely recalled in [ViZu07], says if K_N is ample then Ω_N^1 is polystable, i.e., a direct sum of stable vector bundles of the same slope.

5.6. Lemma. $\text{Hom}(f^*\Omega_N^1, f^*T_N \otimes L^*) \simeq \text{Hom}(\Omega_N^1, T_N \otimes \mathcal{L}^*) = 0$.

Proof. The first isomorphism is the projection formula. The bundle Ω_N^1 is semistable. By [Ko87], 7.11., any map $\Omega_N^1 \rightarrow T_N \otimes \mathcal{L}^*$ vanishes identically as soon as $\mu(\Omega_N^1) > \mu(T_N \otimes \mathcal{L}^*)$.

$$\mu(T_N \otimes \mathcal{L}^*) = \frac{(-K_N - n\mathcal{L}) \cdot K_N^{n-1}}{n} = -\mu(\Omega_N^1) - \mathcal{L} \cdot K_N^{n-1}$$

and (5.4) gives $\mu(T_N \otimes \mathcal{L}^*) = -\mu(\Omega_N^1) + \frac{2n}{m}\mu(\Omega_N^1) + \frac{2(m-n)}{m}\mu(E^{1,0})$. We therefore have to show $\mu(\Omega_N^1) - \mu(T_N \otimes \mathcal{L}^*) = \frac{2(m-n)}{m}(\mu(\Omega_N^1) - \mu(E^{1,0})) > 0$. Because of (5.1) we have to show $\mu(\Omega_N^1) - \mu(E^{1,0}) > 0$.

In the case $\mu(E^{1,0}) = 0$ this follows from $\mu(\Omega_N^1) > 0$ ($n > 0$). In the case $\mu(E^{1,0}) > 0$ this follows from $2\mu(E^{1,0}) \leq \mu(\Omega_N^1)$ ([ViZu07], Theorem 1 and Remark 2). \square

5.2. Holomorphic projective connections. The manifold N_n is said to carry a *holomorphic (normal) projective connection*, if the normalized Atiyah class of Ω_N^1 has the form

$$a(\Omega_N^1) = \frac{c_1(K_N)}{n+1} \otimes id + id \otimes \frac{c_1(K_N)}{n+1}$$

in $H^1(N, \Omega_N^1 \otimes T_N \otimes \Omega_N^1)$, where tensors are identified as before. The following result in the Kähler–Einstein case is due to Kobayashi and Ochiai ([KoOc80]):

5.7. Theorem. *Let N_n be Kähler–Einstein. Then N carries a holomorphic projective connection if and only if N is*

- 1.) $\simeq \mathbb{P}_n(\mathbb{C})$ or
- 2.) a finite étale quotient of a torus or
- 3.) a ball quotient.

Here a ball quotient means the universal covering space of N can be identified with $\mathbb{B}_n(\mathbb{C}) = \{z \in \mathbb{C}^{n+1} \mid |z| < 1\}$, the non compact dual of $\mathbb{P}_n(\mathbb{C})$ in the sense of hermitian symmetric spaces. Then $\pi_1(N)$ acts like a subgroup of $SU(1, n) \simeq \text{HAut}(\mathbb{B}_n(\mathbb{C}))$.

5.8. Lemma. N is a ball quotient and $K_N \equiv -\frac{n+1}{2}\mathcal{L}$.

Proof. Again identify N and $s(N)$, where s denotes the section of $M \rightarrow N$. Then

$$(5.9) \quad 0 \longrightarrow \Omega_N^1 \xrightarrow[\text{df}|_N]{ds} \Omega_M^1|_N \longrightarrow E^{1,0}|_N \longrightarrow 0$$

splits holomorphically. The Atiyah class of a direct sum is the sum of the Atiyah classes. We may therefore compute $a(\Omega_N^1)$ by restricting $a(\Omega_M^1)$ to $N = s(N)$ and to the direct summand Ω_N^1 .

The Atiyah class of Ω_M^1 is given by Proposition 2.2. Projection of the first two summands simply yields

$$(5.10) \quad \frac{c_1(K_M|_N)}{m} \otimes id + id \otimes \frac{c_1(K_M|_N)}{m}$$

in $H^1(N, \Omega_N^1 \otimes T_N \otimes \Omega_N^1)$. We claim that the third summand $g \otimes g^{-1}(\frac{c_1(K_M)}{m})$ vanishes under projection. The third summand goes to the tensor product of g under $H^0(M, S^2\Omega_M^1 \otimes L) \rightarrow H^0(N, S^2\Omega_N^1 \otimes \mathcal{L})$ and the image of $g^{-1}(\frac{c_1(K_M)}{m}) \in H^1(M, T_M \otimes L^*)$ under $H^1(M, T_M \otimes L^*) \rightarrow H^1(N, T_N \otimes \mathcal{L}^*)$. We claim that this last factor vanishes.

We have $K_M = f^*H$ for some $H \in \text{Pic}(N)$. The image of $g^{-1}(\frac{c_1(K_M)}{m})$ under $H^1(M, T_M \otimes L^*) \rightarrow H^1(N, T_N \otimes \mathcal{L}^*)$ is the image of $c_1(H)$ under H^1 of

$$\Omega_N^1 \xrightarrow{\text{df}|_N} \Omega_M^1|_N \xrightarrow{(g|_N)^{-1}} T_M \otimes L^*|_N \xrightarrow{ds^t} T_N \otimes \mathcal{L}^*.$$

By Lemma 5.6, this map vanishes identically. Then the third summand indeed vanishes identically. We find that $a(\Omega_N^1)$ is given by (5.10). The trace gives

$$K_N \equiv \frac{n+1}{m} K_M|_N \quad \text{or} \quad \frac{K_N}{n+1} \equiv \frac{K_M|_N}{m}.$$

Replace this in (5.10) to see that N has a projective connection. Then K_N ample implies N is a ball quotient by Theorem 5.7. Finally $\frac{K_N}{n+1} \equiv \frac{K_M|_N}{m}$ and (5.4) gives $K_N \equiv -\frac{n+1}{2}\mathcal{L}$. \square

Lemma 5.6 implies that g^{-1} induces a commuting diagram

$$(5.11) \quad \begin{array}{ccccccc} 0 & \longrightarrow & f^*\Omega_N^1 & \longrightarrow & \Omega_M^1 & \longrightarrow & f^*E^{1,0} \longrightarrow 0 \\ & & \downarrow \varphi & & \downarrow g^{-1} & & \downarrow \psi \\ 0 & \longrightarrow & f^*E^{1,0*} \otimes L^* & \longrightarrow & T_M \otimes L^* & \longrightarrow & f^*T_N \otimes L^* \longrightarrow 0. \end{array}$$

Here φ is a bundle injection while ψ surjective. By the projection formula we have $\text{Hom}(f^*\Omega_N^1, f^*(E^{1,0*} \otimes \mathcal{L}^*)) \simeq \text{Hom}(\Omega_N^1, E^{1,0*} \otimes \mathcal{L}^*)$ implying that φ is the pullback of some bundle injection $\Omega_N^1 \rightarrow E^{1,0*} \otimes \mathcal{L}^*$. Denote the cocernel by R . Then

$$(5.12) \quad 0 \longrightarrow \Omega_N^1 \longrightarrow E^{1,0*} \otimes \mathcal{L}^* \longrightarrow R \longrightarrow 0.$$

In fact $R = 0$:

5.13. Lemma. $E^{1,0} \simeq T_N \otimes \mathcal{L}^*$ and $m = 2n$.

Proof. Assume $R \neq 0$ or, equivalently, $\text{rk} R = m - 2n > 0$. By (5.12) and (5.4)

$$\det R \equiv -K_{M/N}|_N - (m - n)\mathcal{L} - K_N \equiv \frac{m - 2n}{m}(K_{M/N}|_N + K_N).$$

Then $\mu(R) = \frac{\det R \cdot K_N^{n-1}}{m-2n} = \frac{1}{m}(K_{M/N}|_N + K_N) \cdot K_N^{n-1} = \frac{1}{m}((m - n)\mu(E^{1,0}) + n\mu(\Omega_N^1))$. As $\mu(E^{1,0}) \geq 0$ and $\mu(\Omega_N^1) - \mu(E^{1,0}) > 0$ (see the proof of Lemma 5.6) we find $\mu(R) > 0$.

Consider f_* of (5.11). Let $\mathcal{K} = \ker\{\theta^{1,0} : E^{1,0} \rightarrow E^{0,1} \otimes \Omega_N^1\}$ and $\mathcal{K}' = \ker\{T_N \rightarrow E^{1,0*} \otimes E^{0,1}\}$. We obtain

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \Omega_N^1 & \longrightarrow & f_*\Omega_M^1 & \longrightarrow & \mathcal{K} \longrightarrow 0 \\ & & \downarrow & & \downarrow \simeq & & \\ 0 & \longrightarrow & E^{1,0*} \otimes \mathcal{L}^* & \longrightarrow & f_*T_M \otimes \mathcal{L}^* & \longrightarrow & \mathcal{K}' \otimes \mathcal{L}^* \longrightarrow 0 \\ & & \downarrow & & & & \\ & & R & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

We have an induced surjective map $\mathcal{K} \rightarrow \mathcal{K}' \otimes \mathcal{L}$. By the snake Lemma the kernel is isomorphic to R . This gives an inclusion of Higgs bundles

$$(R, 0) \subset (\mathcal{K}, 0) \subset (E^{1,0} \oplus E^{0,1}, \theta).$$

The latter is semistable as a Higgs bundle. Then $\mu(R) \leq \mu(E^{1,0} \oplus E^{0,1}) = 0$, a contradiction. \square

Proof of Proposition 5.2. Let $f : M_m \rightarrow N_n$ be as in Proposition 5.2. By Lemma 5.13, we have $m = 2n$ and $E^{1,0} \simeq T_N \otimes \mathcal{L}^*$ where $\mathcal{L} \in \text{Pic}(N)$ and $f^*\mathcal{L} = L$. By Lemma 5.6 and Lemma 5.13, the conformal structure g is compatible with

$$0 \longrightarrow f^*\Omega_N^1 \longrightarrow \Omega_M^1 \longrightarrow f^*(T_N \otimes \mathcal{L}) \longrightarrow 0.$$

By Lemma 5.8, N is a ball quotient. Then Ω_N^1 is stable with respect to K_N and $\text{Hom}(\Omega_N^1, \Omega_N^1) \simeq \mathbb{C}$. By Lemma 2.7, the splitting class $e \in H^1(M, f^*(\Omega_N^1 \otimes \Omega_N^1 \otimes \mathcal{L}))$ of the above sequence is alternating on f^*T_N .

The beginning of Leray spectral gives a map

$$H^1(M, f^*(\Omega_N^1 \otimes E^{1,0*})) \longrightarrow H^0(N, R^1 f_* f^*(\Omega_N^1 \otimes E^{1,0*})) \simeq H^0(N, \Omega_N^1 \otimes E^{1,0*} \otimes E^{1,0*}).$$

It sends e to the Kodaira Spencer map δ . It is well known, or follows from the construction of the moduli space \mathcal{A}_{m-n} of abelian varieties as the dual of the Lagrangian Grassmanian, respectively, that δ is symmetric on $E^{1,0}$.

Because of Lemma 5.13 we have

$$\delta \in H^0(N, \Omega_N^1 \otimes E^{1,0*} \otimes E^{1,0*}) \simeq H^0(N, \Omega_N^1 \otimes \Omega_N^1 \otimes \Omega_N^1 \otimes \mathcal{L}^{\otimes 2}).$$

We conclude that δ as a map $T_n^3 \rightarrow \mathcal{L}^{\otimes 2}$ is alternating on the first two tensors and symmetric on the latter two. An easy exercise shows $\delta = 0$.

After an étale base change we may assume that f is induced by a map $N \rightarrow \mathcal{A}_{m-n}$ which must now be constant. Then $\det E^{1,0} \equiv 0$ and by Lemma 5.13, $K_N \equiv -n\mathcal{L}$. Combined with $K_N \equiv -\frac{n+1}{2}\mathcal{L}$ from Lemma 5.8 we find $(\frac{1}{n} - \frac{2}{n+1})K_N \equiv 0$ and K_N implies $\frac{1}{n} - \frac{2}{n+1} = 0$. Then $n = 1$.

By Lemma 5.13, M is a projective surface. It comes with an elliptic fiber bundle $M \rightarrow N$ with typical fiber $\simeq F$ an elliptic curve. The map has a section and the tangent sequence (5.5) splits holomorphically because e is alternating and must vanish. Then M is induced by a representation $\pi_1(N) \rightarrow \text{Aut}(F)$. \square

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